Solution of linear Volterra integro-differential equations of second kind using Shehu transform

Hayriye Bozburun[1] and Haldun Alpaslan Peker[2]

[1]Graduate School of Natural and Applied Sciences, Selçuk University, Turkey
[2]Department of Mathematics, Faculty of Science, Selçuk University, Turkey
E-mail: hay4292@gmail.com[1], pekera@gmail.com[2]

Abstract

Integro-differential equation is an equation where both differential and integral operators appeared together in the same equation. These equations have found extensive applications in applied sciences since it was established by Volterra. Integro-differential equations arise in many mathematical modelling of real life problems such as glass forming process, diffusion process, neutron diffusion problems, heat transfer problems, nanohydrodynamics, epidemiology, circuit analysis. A special class of these equations are the Volterra type which have been used to model heat and mass diffusion processes, biological species coexisting together with increasing and decreasing rate of growth, electromagnetic theory and ocean circulations, among others.

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1 Introduction

Integro-differential equation is an equation where both differential and integral operators appeared together in the same equation. These equations have found extensive applications in applied sciences since it was established by Volterra for the first time in the early 1900 [16, 21]. Integro-differential equations arise in many mathematical modelling of real life problems such as glass forming process, diffusion process in general, neutron diffusion problems, heat transfer problems, nanohydrodynamics, epidemiology, circuit analysis and biological species coexisting together with increasing and decreasing rates of generating [10, 18, 19, 21]. A special class of these equations are the Volterra type which have been used to model heat and mass diffusion processes, biological species coexisting together with increasing and decreasing rate of growth, electromagnetic theory and ocean circulations, among others [9, 18, 21]. The Volterra integro-differential equations were introduced as an immediate consequence of the Volterra's research on the topic of hereditary influences when he was examining a population growth model. This result was termed as Volterra integro-differential equations and these type of equations divided into two groups referred to as first and second kind [9, 10, 15, 19, 20]. Linear Volterra integro-differential equations of the second kind has the following form

$$v^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)v(t)\,dt \tag{1.1}$$

where $v^{(n)}(x)$ denotes the nth derivative of $v(x)$ with respect to x. $f(x)$ is known real-valued function, λ is a non-zero parameter and $K(x,t)$ is the kernel of the equation. In the integro-differential equations, there exists the unknown function $v(x)$ and at least one of its derivatives

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such as $v'(x), v''(x), \ldots$ in the outer and inner part of the integral sign as well. As a result of this situation, the initial conditions should be given to determine a solution for the Volterra integro-differential equation. In order to obtain the integration constants, the initial constants are needed [20, 21].
Solution methods of the differential equations have a great importance in scientific computing and these methods vary in a wide perspective [1, 22] not only solely but also hybridly especially with Padé approximation [7] which is related to continued fractions having many interesting applications in number theory [13, 17].
A Laplace-type integral transform called as the Shehu transform is a generalization of the Laplace and the Sumudu integral transforms for solving differential equations in the time domain. The Shehu transform is denoted by an operator $\mathbb{S}[.]$. The Shehu transform of the function $v(t)$ of exponential order is defined over the set of functions,

$$A = \left\{ v(t) \colon \exists N, \eta_1, \eta_2 > 0,\ |v(t)| < Ne^{\left|\frac{t}{\eta_k}\right|},\ \text{if } t \in (-1)^k \times [0, \infty) \right\} \tag{1.2}$$

by the following integral

$$\mathbb{S}[v(t)] = V(s,u) = \int_0^{\infty} exp\left(\frac{-st}{u}\right) v(t)\, dt \tag{1.3}$$

where $s > 0$, $u > 0$ [14]. If the function $v(t)$ is piecewise continuous in every finite interval $0 \leq t \leq \beta$ and of exponential order α for $t > \beta$, then its Shehu transform $V(s,u)$ exists. These conditions are the sufficient conditions for the existence of Shehu transform of the function $v(t)$ for $t \geq 0$.
Aboodh et al. [2] used Aboodh and double Aboodh transform for solving partial integro-differential equations. Gupta et al. [11] applied Kamal transform for solving linear partial integro-differential equations. Kumar et al. [12] used Mohand transform for solving linear Volterra integro-differential equations. Solution of linear Volterra integro-differential equations of second kind using Aboodh, Mahgoub and Kamal transforms was given by Aggarwal et al. [3, 4, 5]. Aggarwal et al. [6] gave an application of Shehu transform for handling Volterra integral equations of first kind.
The main objective of this study is to determine the solution of linear Volterra integral equations of second kind using Shehu transform and is to apply on some examples to show the efficieny, simplicity and high accuracy of the proposed method without dense computational work [8].

2 Some useful properties of Shehu transform

2.1 Linearity property

Let the functions $\alpha v(t)$ and $\beta w(t)$ be in set A for some non-zero arbitrary constant α and β. Then $\alpha v(t) + \beta w(t)$ is in set A and the following holds [14]:

$$\mathbb{S}[\alpha v(t) + \beta w(t)] = \alpha \mathbb{S}[v(t)] + \beta \mathbb{S}[w(t)]. \tag{2.1}$$

2.2 Change of scale property

Let the function $v(\beta t)$ be in set A for some non-zero arbitrary constants β. Then the following is valid [14]:

$$\mathbb{S}[v(\beta t)] = \frac{u}{\beta} V\left(\frac{s}{\beta}, u\right). \tag{2.2}$$

2.3 Shehu transform of the derivatives of a function

Let the function $v(t)$ be in set A. Then

$$\mathbb{S}[v^{(n)}(t)] = \frac{s^n}{u^n}V(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-(k+1)} v^{(k)}(0) \tag{2.3}$$

where $v^{(n)}(t)$ is the nth derivative of the function $v(t)$ [14].
If $n = 1$, then

$$\mathbb{S}[v'(t)] = \frac{s}{u}V(s,u) - v(0). \tag{2.4}$$

If $n = 2$, then

$$\mathbb{S}[v''(t)] = \frac{s^2}{u^2}V(s,u) - \frac{s}{u}v(0) - v'(0). \tag{2.5}$$

2.4 Shehu transform of some fundamental functions

In this section, Shehu transform of some basic functions is given in the below table [14]. It is important to note that all the functions are in set A.

Table 1: Shehu Transform of Some Fundamental Functions

$v(t)$	$\mathbb{S}[v(t)]$
1	$\frac{u}{s}$
t	$\left(\frac{u}{s}\right)^2$
$\frac{t^n}{n!}$, $n = 0, 1, 2, \ldots$	$\left(\frac{u}{s}\right)^{n+1}$
$\frac{t^n}{\Gamma(n+1)}$, $n = 0, 1, 2, \ldots$	$\left(\frac{u}{s}\right)^{n+1}$
$\exp(\alpha t)$	$\frac{u}{s-\alpha u}$
$t\exp(\alpha t)$	$\left(\frac{u}{s-\alpha u}\right)^2$
$\frac{\sin(\alpha t)}{\alpha}$	$\frac{u^2}{s^2+\alpha^2u^2}$
$\cos(\alpha t)$	$\frac{us}{s^2+\alpha^2u^2}$
$\frac{\sinh(\alpha t)}{\alpha}$	$\frac{u^2}{s^2-\alpha^2u^2}$
$\cosh(\alpha t)$	$\frac{us}{s^2-\alpha^2u^2}$

3 Convolution theorem for Shehu transform

Let the functions $v(t)$ and $w(t)$ be in set A, having Shehu transforms $V(s,u)$ and $W(s,u)$, respectively, i.e. $\mathbb{S}[v(t)] = V(s,u)$ and $\mathbb{S}[w(t)] = W(s,u)$. Then the Shehu transform of the convolution of $v(t)$ and $w(t)$

$$(v * w)(t) = \int_0^\infty v(t)w(t-\tau)d\tau, \tag{3.1}$$

is given [14] by

$$\mathbb{S}[(v * w)(t)] = V(s,u)W(s,u). \tag{3.2}$$

4 Inverse Shehu transform

If $\mathbb{S}[v(t)] = V(s,u)$ then $v(t)$ is called as the inverse Shehu transform of $V(s,u)$ and it is defined by

$$v(t) = \mathbb{S}^{-1}[V(s,u)] \tag{4.1}$$

where $\mathbb{S}^{-1}$ is the inverse Shehu transform operator.

5 Inverse Shehu transform of some fundamental functions

In this section, inverse Shehu transform of some basic functions is given in the below table [14].

Table 2: Inverse Shehu Transform of Some Fundamental Functions

$V(s,u)$	$\mathbb{S}^{-1}[V(s,u)] = v(t)$
$\frac{u}{s}$	1
$\left(\frac{u}{s}\right)^2$	t
$\left(\frac{u}{s}\right)^{n+1}$	$\frac{t^n}{n!},\ n = 0,1,2,\ldots$
$\left(\frac{u}{s}\right)^{n+1}$	$\frac{t^n}{\Gamma(n+1)},\ n = 0,1,2,\ldots$
$\frac{u}{s-\alpha u}$	$\exp(\alpha t)$
$\frac{u^2}{s^2+\alpha^2u^2}$	$\frac{\sin(\alpha t)}{\alpha}$
$\frac{us}{s^2+\alpha^2u^2}$	$\cos(\alpha t)$
$\frac{u^2}{s^2-\alpha^2u^2}$	$\frac{\sinh(\alpha t)}{\alpha}$
$\frac{us}{s^2-\alpha^2u^2}$	$\cosh(\alpha t)$

6 Shehu transform for linear Volterra integro-differential equations of second kind

In this section, we present Shehu transform for solving linear Volterra integro-differential equations of second kind given as

$$v^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)v(t)\,dt \tag{6.1}$$

in which, for this work, we will assume that the kernel $K(x,t)$ is a difference kernel that can be expressed by difference $(x-t)$. Hence, the above linear Volterra integro-differential equation of second kind can be expressed as follows

$$v^{(n)}(x) = f(x) + \lambda \int_0^x K(x-t)v(t)\,dt \tag{6.2}$$

with the initial conditions given by

$$v(0) = a_0,\ v'(0) = a_1,\ \ldots\ ,\ v^{(n-1)}(0) = a_{n-1}. \tag{6.3}$$

Applying the Shehu transform to both sides of the last equation gives the following expression:

$$\left(\frac{s}{u}\right)^n \mathbb{S}[v(x)] = \left(\frac{s}{u}\right)^{n-1} a_0 + \left(\frac{s}{u}\right)^{n-2} a_1 + \ldots + a_{n-1} + \mathbb{S}[f(x)] + \lambda\, \mathbb{S}\left[\int_0^x K(x-t)v(t)\,dt\right]. \tag{6.4}$$

Using convolution theorem for Shehu transform and rearranging the expression,

$$\mathbb{S}[v(x)] = \left(\frac{u}{s}\right) a_0 + \left(\frac{u}{s}\right)^2 a_1 + \ldots + \left(\frac{u}{s}\right)^n a_{n-1} + \left(\frac{u}{s}\right)^n \mathbb{S}[f(x)] + \left(\frac{u}{s}\right)^n \lambda\, \mathbb{S}[K(x)]\, \mathbb{S}[v(x)] \tag{6.5}$$

is obtained.
Applying inverse Shehu transform on both sides of this equation gives the following

$$v(x) = a_0 + a_1 x + \ldots + a_{n-1}\frac{x^{n-1}}{(n-1)!} + \mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^n \mathbb{S}[f(x)]\right] + \mathbb{S}^{-1}\left[\left(\frac{u}{s}\right)^n \lambda\, \mathbb{S}[K(x)]\, \mathbb{S}[v(x)]\right] \tag{6.6}$$

which is the required solution of the linear Volterra integro-differential equation of second kind considered at the beginning.

7 Applications

In this part, some applications are presented to illustrate the effectiveness of Shehu transform for solving the linear Volterra integro-differential equation of second kind.

Application 1: Consider the linear Volterra integro-differential equation of second kind given by

$$v'(x) = 3 + \int_0^x v(t)dt \tag{7.1}$$

with the initial condition $v(0) = 3$.
Applying the Shehu transform to both sides of the given equation and using initial condition, the following is found:

$$\left(\frac{s}{u}\right) \mathbb{S}[v(x)] - v(0) = \mathbb{S}[3] + \mathbb{S}\left[\int_0^x v(t)dt\right]. \tag{7.2}$$

Then, this gives

$$\mathbb{S}[v(x)] = 3\left(\frac{u}{s}\right) + 3\left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)\, \mathbb{S}\left[\int_0^x v(t)dt\right]. \tag{7.3}$$

Using convolution theorem for Shehu transform and rearranging the expression, the following

$$\mathbb{S}[v(x)] = 3\left(\frac{u}{s}\right) + 3\left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)\, \mathbb{S}[1]\,\mathbb{S}[v(x)] \tag{7.4}$$

$$\Rightarrow \left(1 - \left(\frac{u}{s}\right)^2\right)\mathbb{S}[v(x)] = 3\left(\frac{u}{s}\right) + 3\left(\frac{u}{s}\right)^2 \tag{7.5}$$

is obtained. Consequently, we have

$$\mathbb{S}[v(x)] = 3\left(\frac{u}{s-u}\right). \tag{7.6}$$

Applying inverse Shehu transform on both sides of this equation gives the following

$$v(x) = \mathbb{S}^{-1}\left[3\left(\frac{u}{s-u}\right)\right] = 3\,\mathbb{S}^{-1}\left[\frac{u}{s-u}\right] = 3e^x \tag{7.7}$$

which is the exact solution of the considered equation.

Application 2: Consider linear Volterra integro-differential equation of second kind given as

$$v''(x) = x + \int_0^x (x-t)v(t)dt \tag{7.8}$$

with the initial conditions $v(0) = 0$ and $v'(0) = 1$.
Having applied the Shehu transform to both sides of the given equation and using initial conditions, the following is obtained:

$$\left(\frac{s}{u}\right)^2\mathbb{S}[v(x)] - \left(\frac{s}{u}\right)v(0) - v'(0) = \mathbb{S}[x] + \mathbb{S}\left[\int_0^x (x-t)v(t)dt\right]. \tag{7.9}$$

Then, this gives

$$\mathbb{S}[v(x)] = \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^2\, \mathbb{S}\left[\int_0^x (x-t)v(t)dt\right]. \tag{7.10}$$

Using convolution theorem for Shehu transform and rearranging the expression, we have

$$\mathbb{S}[v(x)] = \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^2\, \mathbb{S}[x]\,\mathbb{S}[v(x)] \tag{7.11}$$

$$\Rightarrow \left(1 - \left(\frac{u}{s}\right)^4\right)\mathbb{S}[v(x)] = \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^4 \tag{7.12}$$

$$\Rightarrow \left(1 - \left(\frac{u}{s}\right)^4\right)\mathbb{S}[v(x)] = \left(\frac{u}{s}\right)^2\left(1 + \left(\frac{u}{s}\right)^2\right). \tag{7.13}$$

This implies that

$$\mathbb{S}[v(x)] = \frac{u^2}{s^2 - u^2}. \tag{7.14}$$

Applying inverse Shehu transform on both sides of this equation gives the following

$$v(x) = \mathbb{S}^{-1}\left[\frac{u^2}{s^2 - u^2}\right] = \sinh x \tag{7.15}$$

which is the exact solution of the considered equation.

Application 3: Consider the following linear Volterra integro-differential equation of second kind

$$v''(x) = 1 + x + \int_0^x (x-t)v(t)dt \tag{7.16}$$

with the initial conditions $v(0) = 1$ and $v'(0) = 1$.
Applying the Shehu transform to both sides of the given equation and using initial conditions, the following is obtained:

$$\left(\frac{s}{u}\right)^2 \mathbb{S}[v(x)] - \left(\frac{s}{u}\right) v(0) - v'(0) = \mathbb{S}[1] + \mathbb{S}[x] + \mathbb{S}\left[\int_0^x (x-t)v(t)dt\right]. \tag{7.17}$$

Then, this gives

$$\mathbb{S}[v(x)] = \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^2 \mathbb{S}\left[\int_0^x (x-t)v(t)dt\right]. \tag{7.18}$$

Using convolution theorem for Shehu transform and rearranging the expression, we have

$$\mathbb{S}[v(x)] = \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^2 \mathbb{S}[x]\,\mathbb{S}[v(x)] \tag{7.19}$$

$$\Rightarrow \left(1 - \left(\frac{u}{s}\right)^4\right) \mathbb{S}[v(x)] = \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4. \tag{7.20}$$

This implies that

$$\mathbb{S}[v(x)] = \frac{u}{s - u}. \tag{7.21}$$

Applying inverse Shehu transform on both sides of this equation gives the following

$$v(x) = \mathbb{S}^{-1}\left[\frac{u}{s - u}\right] = e^x \tag{7.22}$$

which is the exact solution of the considered equation.

Application 4: Consider linear Volterra integro-differential equation of second kind given as

$$v'''(x) = -1 + \int_0^x v(t)dt \tag{7.23}$$

with the initial conditions $v(0) = 1,\ v'(0) = 1$ and $v''(0) = -1$.
Having applied the Shehu transform to both sides of the given equation and using initial conditions, the following is obtained:

$$\left(\frac{s}{u}\right)^3 \mathbb{S}[v(x)] - \left(\frac{s}{u}\right)^2 v(0) - \left(\frac{s}{u}\right) v'(0) - v''(0) = -\mathbb{S}[1] + \mathbb{S}\left[\int_0^x v(t)dt\right]. \tag{7.24}$$

Then, this gives

$$\mathbb{S}[v(x)] = \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^3 - \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^3 \mathbb{S}\left[\int_0^x v(t)dt\right]. \tag{7.25}$$

Using convolution theorem for Shehu transform and rearranging the expression, we have

$$\mathbb{S}[v(x)] = \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^3 - \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^3 \mathbb{S}[1]\,\mathbb{S}[v(x)] \tag{7.26}$$

$$\Rightarrow \left(1 - \left(\frac{u}{s}\right)^4\right) \mathbb{S}[v(x)] = \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^3 - \left(\frac{u}{s}\right)^4 \tag{7.27}$$

$$\Rightarrow \left(1 - \left(\frac{u}{s}\right)^4\right) \mathbb{S}[v(x)] = \left(\frac{u}{s}\right)\left(1 - \left(\frac{u}{s}\right)^2\right) + \left(\frac{u}{s}\right)^2 \left(1 - \left(\frac{u}{s}\right)^2\right). \tag{7.28}$$

This implies that

$$\mathbb{S}[v(x)] = \frac{us}{s^2 + u^2} + \frac{u^2}{s^2 + u^2}. \tag{7.29}$$

Applying inverse Shehu transform on both sides of this equation gives the following

$$v(x) = \mathbb{S}^{-1}\left[\frac{us}{s^2 + u^2} + \frac{u^2}{s^2 + u^2}\right] = \mathbb{S}^{-1}\left[\frac{us}{s^2 + u^2}\right] + \mathbb{S}^{-1}\left[\frac{u^2}{s^2 + u^2}\right] = \cos x + \sin x \tag{7.30}$$

which is the exact solution of the considered equation.

Application 5: Consider linear Volterra integro-differential equation of second kind given as

$$v'''(x) = 1 + x - x^2 + \int_0^x (x - t)v(t)dt \tag{7.31}$$

with the initial conditions $v(0) = 3,\ v'(0) = 1$ and $v''(0) = 1$.
Having applied the Shehu transform to both sides of the given equation and using initial conditions, the following is obtained:

$$\left(\frac{s}{u}\right)^3 \mathbb{S}[v(x)] - \left(\frac{s}{u}\right)^2 v(0) - \left(\frac{s}{u}\right) v'(0) - v''(0) = \mathbb{S}[1] + \mathbb{S}[x] - \mathbb{S}[x^2] + \mathbb{S}\left[\int_0^x (x - t)v(t)dt\right]. \tag{7.32}$$

Then, this gives

$$\mathbb{S}[v(x)] = 3\left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^5 - 2\left(\frac{u}{s}\right)^6 + \left(\frac{u}{s}\right)^3 \mathbb{S}\left[\int_0^x (x - t)v(t)dt\right]. \tag{7.33}$$

Using convolution theorem for Shehu transform and rearranging the expression, we have

$$\mathbb{S}[v(x)] = 3\left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^5 - 2\left(\frac{u}{s}\right)^6 + \left(\frac{u}{s}\right)^3 \mathbb{S}[x]\,\mathbb{S}[v(x)] \tag{7.34}$$

$$\Rightarrow \left(1 - \left(\frac{u}{s}\right)^5\right)\mathbb{S}[v(x)] = 3\left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^5 - 2\left(\frac{u}{s}\right)^6 \tag{7.35}$$

$$\Rightarrow \left(1 - \left(\frac{u}{s}\right)^5\right)\mathbb{S}[v(x)] = \left(\frac{u}{s}\right)\left(1 + \left(\frac{u}{s}\right) + \left(\frac{u}{s}\right)^2 + \left(\frac{u}{s}\right)^3 + \left(\frac{u}{s}\right)^4\right) + 2\left(\frac{u}{s}\right)\left(1 - \left(\frac{u}{s}\right)^5\right). \tag{7.36}$$

This implies that

$$\mathbb{S}[v(x)] = \frac{u}{s-u} + 2\,\frac{u}{s}. \tag{7.37}$$

Applying inverse Shehu transform on both sides of this equation gives the following

$$v(x) = \mathbb{S}^{-1}\left[\frac{u}{s-u} + 2\,\frac{u}{s}\right] = \mathbb{S}^{-1}\left[\frac{u}{s-u}\right] + \mathbb{S}^{-1}\left[2\,\frac{u}{s}\right] = e^x + 2 \tag{7.38}$$

which is the exact solution of the considered equation.

8 Conclusion

In this paper, Shehu transform was applied to find the exact solution of linear Volterra integro-differential equations of second kind. The simplicity, efficiency and high accuracy of the Shehu transform without dense computational work are clearly illustrated by the given applications.

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